



## Grade 7/8 Math Circles

March 27/28/29/30, 2023

### Vectors

#### Introduction

When you hear the word “measurement”, what do you think of? Maybe you imagine a ruler with centimetre and inch measurements. Or measuring cups that you use to cook and bake with. Now, when you hear the word “direction”, what do you think of? Maybe the words left, right, north, south, east, or west come to mind. In this lesson, we will look at how we combine measurements and directions in mathematics and physics, using something called vectors!

#### Stop and Think

Think of an instance where you would use both measurement and direction.

#### Defining Scalars and Vectors

A **scalar** is a measurement and a **vector** is a measurement with a direction. Think about walking from one side of the room to the other. Are you just walking or are you walking with a direction? Pretty much every movement you make has a direction and so walking across the room is an example of a vector.

#### Exercise 1

Categorize the following as scalar or vector quantities.

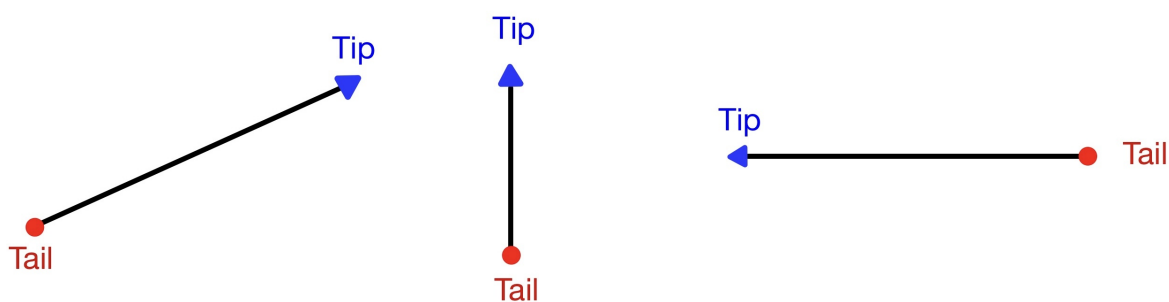
- a) 2 cups of flour being added to a cookie recipe
- b) A thermometer reading  $20^{\circ}$  Celsius
- c) A runner running clockwise around a track
- d) A backyard with an area of  $50 \text{ m}^2$
- e) A car driving 10 km west
- f) A cyclist traveling at 30 km/h



g) A cyclist traveling at 30 km/h north

## Vector Notation

When working with vectors in math, it's often helpful to draw them out. We represent vectors with arrows where the length of the arrow determines the size of the quantity and the way the arrow is pointing determines the direction.



We call the side of the arrow with the arrow head the **tip** and the other side is called the **tail**.

When working with vectors on the Cartesian plane, we write the values of the vector vertically in square brackets like so:

$$\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

The top number (the 2 in the example above) represents the horizontal measurement and the bottom number (the  $-5$  in the example) represents the vertical measurement. Also notice that we name vectors as a variable (normally  $v$ ) with an arrow over top of it.

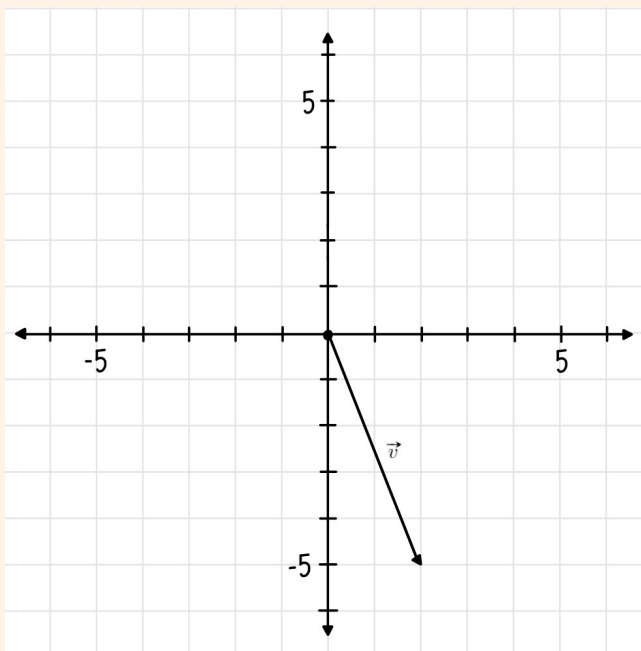
### Example A

Sketch  $\vec{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$  on the Cartesian plane.

The vector  $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$  relates to the coordinate  $(2, -5)$ . If we place the point  $(2, -5)$  on the graph,

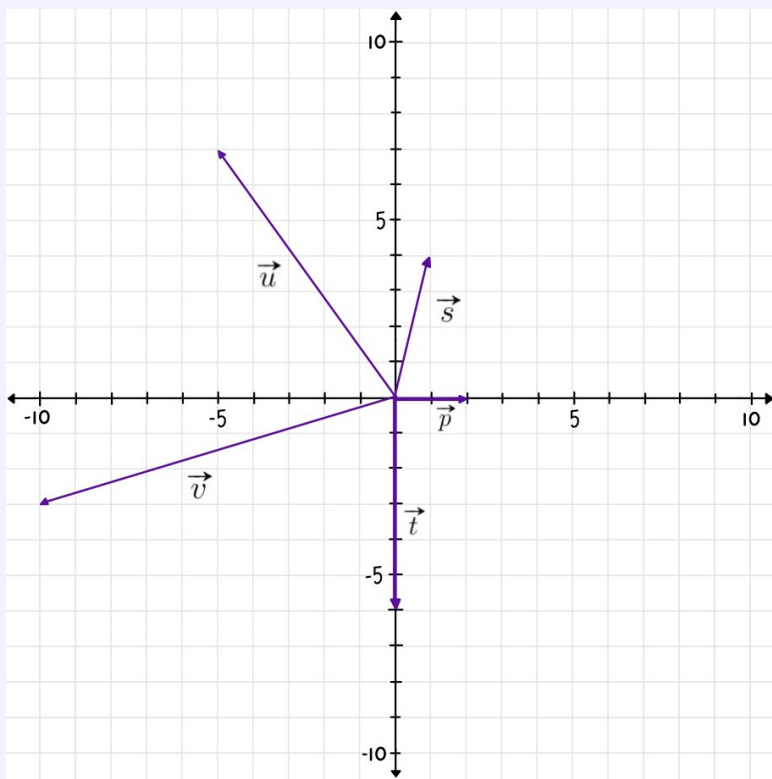


the vector  $\vec{v}$  is the arrow from the origin to  $(2, -5)$ .



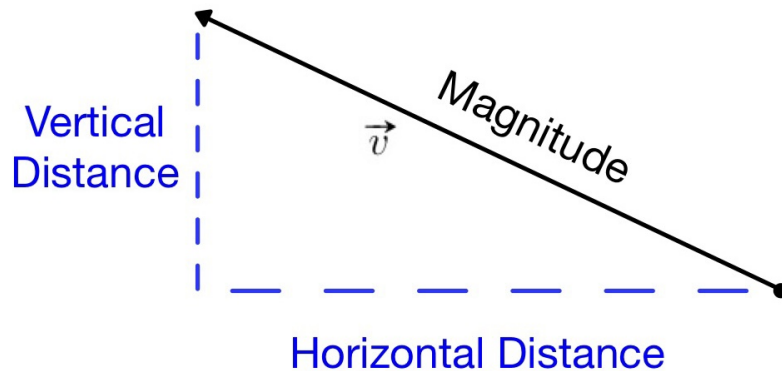
### Exercise 2

Write each of the given vectors in vector notation.



### Vector Magnitude

When we think of the **magnitude** of a vector, we can think of its length. Notice that our vector notation gives us the vertical and horizontal distances, but not the magnitude (or size) of the vector. To calculate magnitude, we can think of a right angle triangle.





Some of you may look at a right triangle and think of trigonometry and the Pythagorean theorem, which is exactly what we use to find magnitude. The formula is as follows:

$$\|\vec{v}\| = \sqrt{x^2 + y^2}$$

Here, the  $x$  denotes horizontal distance and the  $y$  denotes vertical distance. We use the double bars around the vector to denote the magnitude of the vector.

### Exercise 3

Find the magnitude of each vector.

a)  $\vec{a} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$

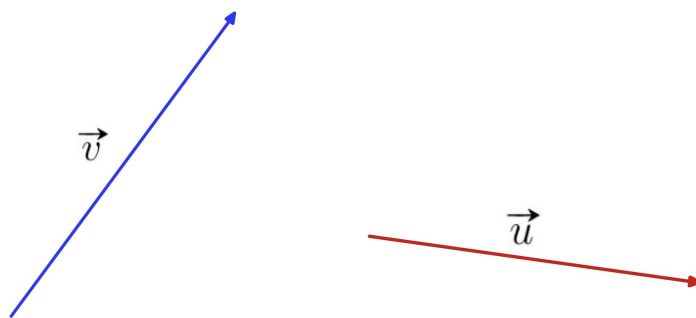
b)  $\vec{b} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

c)  $\vec{c} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

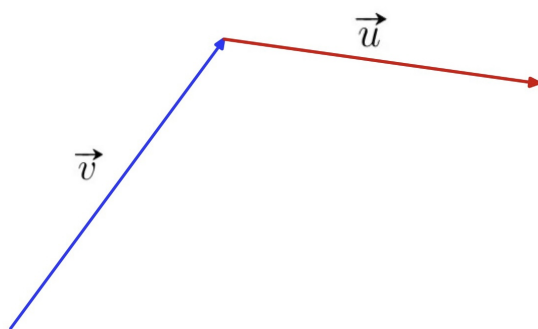
d)  $\vec{d} = \begin{bmatrix} -10 \\ -1 \end{bmatrix}$

## Vector Addition

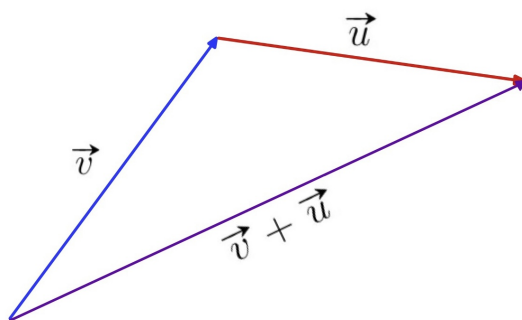
To understand vector addition, we will walk through an example to find  $\vec{v} + \vec{u}$ .



To add  $\vec{v}$  and  $\vec{u}$ , we attach the tail of  $\vec{u}$  to the tip of  $\vec{v}$ .



So,  $\vec{v} + \vec{u}$  is the vector connects the tail of  $\vec{v}$  to the tip of  $\vec{u}$  which should look like this:



### Stop and Think

Do you think  $\vec{v} + \vec{u} = \vec{u} + \vec{v}$ ?

(Note: 2 vectors are equal if they have the same magnitude and direction)

Let's look at how vector addition works algebraically.

### Example B

Calculate  $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ .

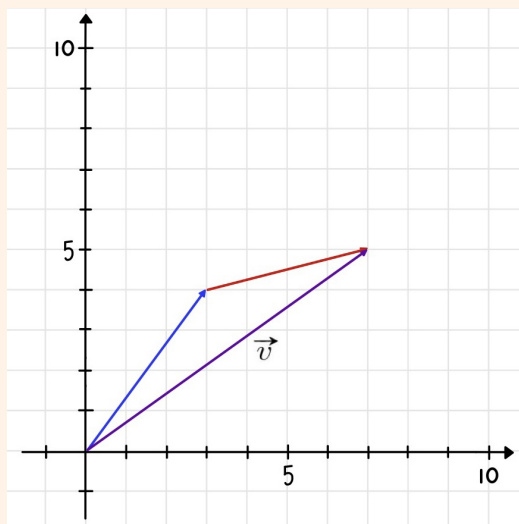
To find  $\vec{v}$ , simply add the horizontal measurements to get the top number of  $\vec{v}$  and add the vertical measurements to get the bottom number of  $\vec{v}$ .



$$\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+4 \\ 4+1 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

So,  $\vec{v} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ .

To confirm this, let's add these vectors on the Cartesian plane.



The vector  $\vec{v}$  ends at the point  $(7,5)$  so we confirm that  $\vec{v} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$ .

#### Exercise 4

Calculate the following vector addition operations.

a)  $\begin{bmatrix} -4 \\ -2 \end{bmatrix} + \begin{bmatrix} 11 \\ -18 \end{bmatrix}$

b)  $\begin{bmatrix} 1 \\ 16 \end{bmatrix} + \begin{bmatrix} -5 \\ 10 \end{bmatrix}$

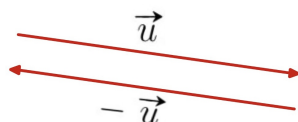
c)  $\begin{bmatrix} 7 \\ 2 \end{bmatrix} + \begin{bmatrix} 12 \\ 15 \end{bmatrix}$

d)  $\begin{bmatrix} 6 \\ -8 \end{bmatrix} + \begin{bmatrix} 14 \\ 28 \end{bmatrix}$

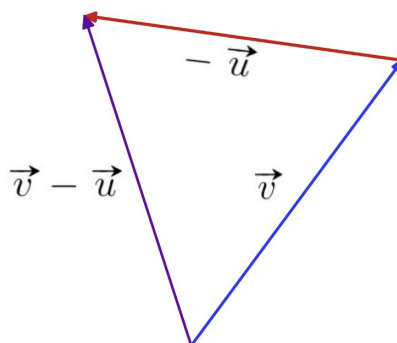


## Vector Subtraction

Given the vector  $\vec{u}$ ,  $-\vec{u}$  has the same magnitude of  $\vec{u}$ , just in the opposite direction.



So, to find  $\vec{v} - \vec{u}$ , we simply perform vector addition with  $\vec{v}$  and  $-\vec{u}$ .



### Example C

Calculate  $\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ .

Similar to addition, to find  $\vec{v}$ , simply subtract the horizontal measurements to get the top number of  $\vec{v}$  and subtract the vertical measurements to get the bottom number of  $\vec{v}$ .

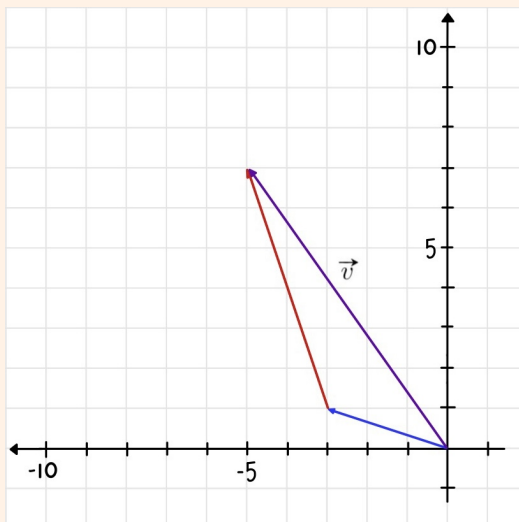
$$\vec{v} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 - 2 \\ 1 - (-6) \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$$

So,  $\vec{v} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ .





To confirm this, let's add these vectors on the Cartesian plane.



The vector  $\vec{v}$  ends at the point  $(-5, 7)$  so we confirm that  $\vec{v} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$ .

### Exercise 5

Calculate the following vector subtraction operations.

a)  $\begin{bmatrix} 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

b)  $\begin{bmatrix} -2 \\ 11 \end{bmatrix} - \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

c)  $\begin{bmatrix} -20 \\ -4 \end{bmatrix} - \begin{bmatrix} -18 \\ -4 \end{bmatrix}$

d)  $\begin{bmatrix} 12 \\ 8 \end{bmatrix} - \begin{bmatrix} 6 \\ -1 \end{bmatrix}$



## Scalar Multiplication

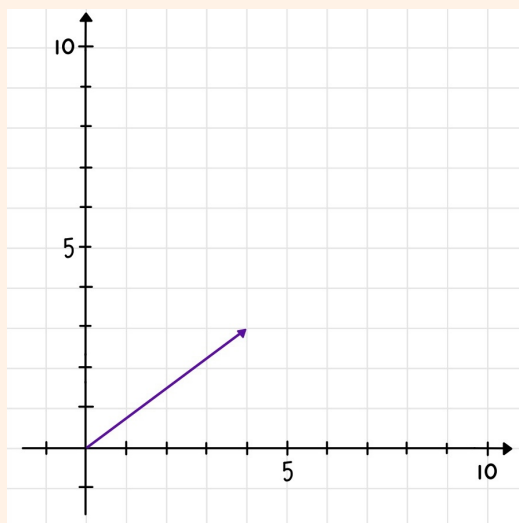
### Stop and Think

What do you think would happen to a vector if we multiplied it by 2? What about 3? What about  $-5$ ?

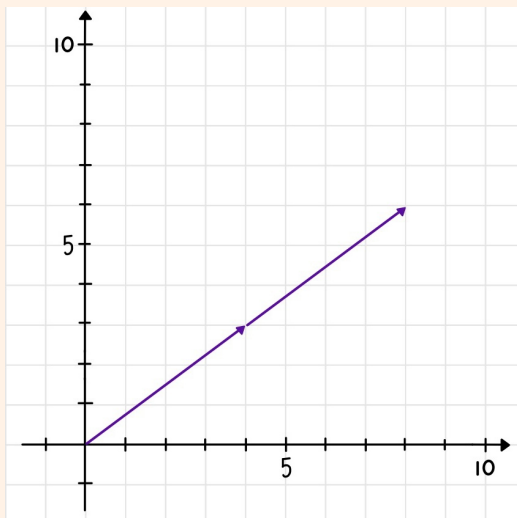
It may (or may not) be intuitive that doubling a vector would double its size (or magnitude). It's the same idea with tripling the vector. We already know multiplying by a negative number reverses the direction of the vector so multiplying by  $-5$  would be 5 times the length of the original vector pointing in the opposite direction.

### Example D

We have the following vector  $\vec{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ .



Find the vector that's double the vector  $\vec{v}$ .



So, the vector  $2\vec{v} = 2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 4 \\ 2 \times 3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$ .

We can see in the example that doubling the magnitude of the vector also doubles the horizontal and vertical measurements. This idea holds for all scalar multiplication. Whenever we multiply a vector by a number, the horizontal and vertical distances are both multiplied by that number.

### Exercise 6

Calculate the following scalar multiplication operations.

a)  $4 \begin{bmatrix} -4 \\ 13 \end{bmatrix}$

b)  $\frac{1}{2} \begin{bmatrix} 17 \\ 4 \end{bmatrix}$

c)  $-3 \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

d)  $-\frac{1}{3} \begin{bmatrix} 15 \\ 0 \end{bmatrix}$



## Putting it All Together

When performing operations on vectors, it's important to follow order of operations (or BEDMAS) like we do with normal numbers. However in our case with vectors, we only have multiplication and addition/subtraction. So we will first perform scalar multiplication and then vector addition/subtraction from left to right.

### Exercise 7

Perform the following vector operations to solve for  $x$  and  $y$ .

$$\text{a) } \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} -5 \\ 8 \end{bmatrix}$$

$$\text{b) } \begin{bmatrix} x \\ y \end{bmatrix} = -3 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + 4 \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$$\text{c) } \begin{bmatrix} -15 \\ -26 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ -2 \end{bmatrix} - 4 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{d) } \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = - \begin{bmatrix} x \\ y \end{bmatrix} + 4 \begin{bmatrix} \frac{1}{3} \\ 2 \end{bmatrix}$$

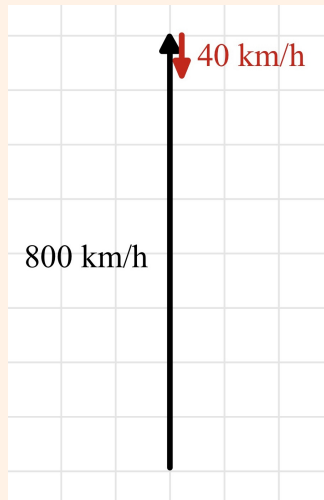
## Word Problems

Word problems are a very important part of learning about vectors. There are many real world problems that we can solve using vectors. We will look at what some of these problems are.

### Example E

On a clear day, a plane is flying north at a speed of 800 km/h. Suddenly, the wind picks up and starts blowing south at a speed of 40 km/h. How fast does the plane fly now, taking the wind into account?

First, sketch out the plane and wind vectors.



We can consider north to be the positive direction and south to be the negative direction. We use the first letter of the direction in square brackets to represent the direction while performing the calculation. Thus the resulting vector would be:

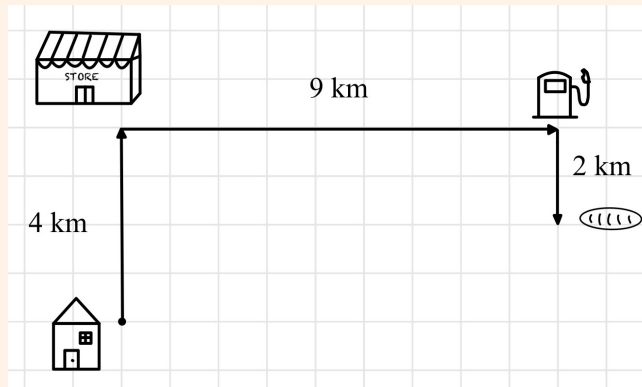
$$\begin{aligned} & 800 \text{ km/h [N]} + 40 \text{ km/h [S]} \\ &= 800 \text{ km/h [N]} + (-40 \text{ km/h [N]}) \\ &= 800 \text{ km/h [N]} - 40 \text{ km/h [N]} \\ &= 760 \text{ km/h [N]} \end{aligned}$$

So the plane is flying 760 km/h north.

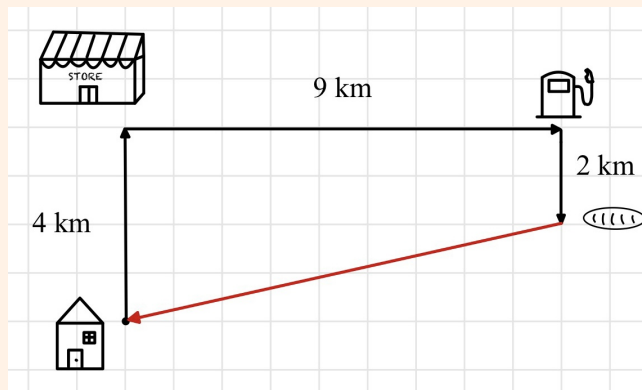
### Example F

Hypatia is running some errands. She starts at her house and then moves 4 km north to go to the grocery store. She then goes to get gas at the gas station which is 9 km east from the grocery store. Finally she picks up some bread at a bakery which is 2 km south of the gas station. How far is Hypatia from her house after completing her errands? What direction does she need to go to get back home?

First, sketch out Hypatia's route.



Next, draw a vector from the bakery to Hypatia's house.



Finally, find the magnitude of the vector from the bakery to Hypatia's house.

The horizontal distance is 9 km and the vertical distance is 2 km. So the magnitude will be:

$$\sqrt{9^2 + 2^2} = \sqrt{81 + 4} = \sqrt{85} \approx 9.22 \text{ km}$$

Hypatia is about 9.22 km from her house and we can see that she needs to travel southwest to get back home.

### Exercise 8

Euler is trying to follow directions to get to his friend's house. The directions are as follows:

- 5 km west
- 2 km south
- 6 km north



- 8 km east
- 3 km north

(Note that Euler's GPS likes to take him on the most inefficient route).

How far is Euler's house from his friend's house?

What would be a more efficient route for Euler to follow?

## Extend Your Learning

In this lesson, we talked about vectors with  $x$  measurements and  $y$  measurements. Because these vectors only have 2 entries, we call them **2-dimensional** vectors. If you decide to take upper level physics or linear algebra classes in the future, you will work with vectors with dimensions greater than 2. Do you think you could visualize what a 3-dimensional vector looks like? What about 4 dimensions? What about 100 dimensions?